An Effective Use of CAS for Reasoning as a Cognitive Tool

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Abstract—*The effective use of CAS (computer algebra system) were introduced and were expected to develop ways in which to use CAS effectively as "tools" for mathematics education and to achieve good results as a teaching method for mathematics education. In the 2000s, with the evolution of software and hardware, it has become easy to use the CAS in classroom. However, the proportion of teachers who use CAS in classroom is still quite low and it is hard to say that CAS has started to be commonly used in classroom. We must consider about the effective use seriously. Then we need to consider "a tool theory" in the cognitive science. Humans use strategies to solve problems. Strategies are used as knowledge to plan solutions and decide procedures. The techniques for theorem prove using CAS is being developed. We must consider the theorem prove from not only the perspective of its effect on cognitive science, but also from the perspective of mathematical studies. We can explain new use possibility of the CAS by being based on the theory of cognitive science.*

Index Terms—*Mathematics education, Computer algebra system, Cognitive science*

1. *INTRODUCTION*

In the 1990s, there were introduced computer -based mathematics education, and the effective use of CAS (computer algebra system) was part of this attempt. These proposals were expected to develop ways in which to use CAS effectively as "tools" for mathematics education and to achieve good results as a teaching method for mathematics education. However, these efforts lacked a clearly defined direction. Researchers were uncertain as to what kind of basic principles the utilization of CAS for mathematics education stood for, or what goals we were trying to achieve.

In the 2000s, with the evolution of software and hardware, it has become easy to use the CAS in classes. However, the proportion of teachers who use CAS in classroom is still quite low and it is hard to say that CAS has started to be commonly used in classroom. There is the fact that the value of CAS as a teaching tool is not recognized.

In mathematics education, what is the purpose of using CAS? That is to assisting the developing of students' mathematical thinking. In particular, the CAS has big possibility in learning of the mathematics. However, we did not perform a clear study about its possibility. It is because we thought the CAS to be the expert engineers' tool. The present condition has changed. We must consider about the effective use seriously. Then we need to consider "a tool theory" in the cognitive science. We can explain new use possibility of the CAS by being based on the theory of cognitive science.

According to the three-level human behavior model of Rasmussen, automatic human actions can be classified into the three levels of skill-, rule- and knowledge-based actions(Fig.2.1, [9]).

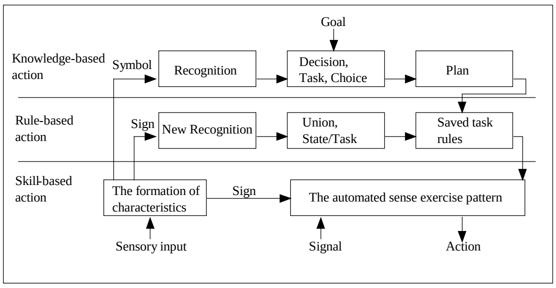

Figure 2.1 Three-Level Model of Human Action

A skill-based action is a response that occurs in less than 1 second ([7]). A chain of skill-based actions is a rule-based action. Thinking about how to solve a problem is a knowledge-based action.

Skill-based actions are performed smoothly without intentional control. Rule-based actions require a great deal of repetitive practice in order to be transferred to the skill-based level. First, the external conditions must be recognized, then the rules for composing the act are combined with the conditions required to carry out the behavior. Knowledge-based actions require the recognition of external conditions, the interpretation of these conditions, the construction of a psychological model for considering solutions, planning, and finally, the use of the other two behavior levels to carry out the action.

This is a process model in which mastery of behavior requiring thought is internalized to the point where it can be carried out unconsciously. Mistakes can be explained as omitted steps, or for example, as pushing the wrong nearby button in smoothly carried out skill-based actions. In the case of knowledge-based actions, illusion can lead to error. In the present study, this process was analyzed using Rasmussen's three-level human behavior model in order to identify what functions are essential to facilitating smooth

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action and learning. Behavior used to learn about problems and how to solve them is classified in detail according to the three-level model. Humans act by classifying issues and their relationships by consciously combining them. Humans control themselves by constantly observing, thinking about, evaluating, and integrating their behavior in order to achieve accuracy, continuity, consistency, and normality ([4]).

2. Cognitive Science in Learning

2.1. Norman's theory

CAS can be used as effective tools for calculation and mathematical thinking. CAS is a kind of external tool. An external tool can help a person promote the use of internal tools and thus improve the efficiency of overall use of internal and external tools ([6]). The effective use of CAS can play an important role when a person is trying to understand new concepts. As an external tool, CAS was developed to actually save a lot of time and effort in making calculations and enable faster manipulation of mathematical formulae. Although modern CAS is named algebra systems, it actually has much more performances than symbolic manipulation. With these performances, CAS has become even more powerful tools for mathematics. If we hope to utilize CAS in mathematics education, we need to find some new methods of teaching (based on the accomplishments obtained from cognitive science) that differ from the traditional methods of using "paper and pencil".

Careful consideration of the aspects of the science as an "external tool" is important for bringing about changes in approaches to the problem. CAS as the external tools substitute for some human abilities and thus bring a new potential. At the same time, CAS requires different kinds of abilities and knowledge. When we consider the effects of mathematics education utilizing CAS, clarifying what kinds of abilities and knowledge are required is crucial.

A cognitive tool is a tool for embodying the image of an outer object that appears in the consciousness on the basis of the human perception. Norman divided the human cognition, when using technology, into two categories ([5]).

- Experimental cognition: to cope without conscious to the outside world change.
- Reflective cognition: to deeply understand for thinking of the meaning of each thing and referring back to experience.

In a learning environment using CAS, the human cognition is divided into experiential one and introspective one. The tools for the experiential cognition have to be able to exploit a rich sensory stimulus. The tools for introspective cognition must be supporting the search of ideas. This cognition needs a different support. It does not function well if we give experiential cognition an introspective tool. Vice versa also will not work. Therefore, teachers must distinguish which type of cognition the tools are supporting. And for that purpose, the teachers must provide tools that offer appropriate support for a certain activity. In order to use effectively technology, the teachers must consider which one of the two cognitions is suitable for its learning activity. If the consideration for the cognition mode is neglected, the effective use is impossible. If learners enjoy the experiences when they must do introspective searches, they misunderstand its activity as introspective activity.

According to Norman, there are three learning categories that are useful in determining which of the two cognition modes is suitable for its learning activity:

- Accretion: to accumulate facts.
- Tuning: to adjust it in a way to use skills involving introspection with an experiential mode.
- Restructuring: to form an appropriate conceptual structure by introspection.

In many cases, accretion and tuning are seen as experiential modes, restructuring is seen as an introspective mode. To make CAS appropriately work as a cognitive tool, teachers must determine what responds to a certain learning activity of students.

2.2. Strategies

Strategies are used as knowledge to plan solutions and decide procedures. When these procedures, in general or for the most part, obtain the correct answer, the procedure is called a heuristic; however, such heuristics do not always result in a correct solution.

Strategies are used even when human beings solve mathematical problems. Recognition knowledge and experience are used as "doing it like this is effective in this case". The ability to rapidly reference knowledge is required for strategies based on experience. The famous book by the mathematician Polya, "How to solve it"([8]), showed the processes of mathematical problem solving; however, one can not learn how to use heuristics in problem solving just by reading a book.

In researching problem solving, there are two contrasting concepts. The first emphasizes insight, flash, and senses, while the second emphasizes experiential knowledge. The former concept employs a strong tendency to perceive that strategies of thought are learned through the experience of problem solving. In other words, it is assumed that an intuitive feelings and specific technical abilities can be acquired. In the latter concept, it is assumed that problem solving ability arises from the accumulation of rules inherent to the domain provided by an individual problem.

Such differences depend on the problem's nature, domain, and level, and the type of person involved in the learning process. In addition, it is difficult to establish clear boundary lines between these two concepts. In problem solving, experiential knowledge plays a large role. Heuristics are general ideas or algorithms (a procedure providing the correct solution), and are widely used. Heuristics are equal to "the logic of a thought".

Examples of extremely general strategies are "try to draw a figure if you come across a difficult problem", and "search for similar problems that you have experience with". There are also concrete strategies we are familiar with, such as "A problem requiring the comparison of quantities requires two differences, and a transform formula" and "try to make clauses that differ next to each other for number sum sequence problems" ([2]).

3. *The Effect of the Theorem Prover*

As a representative of a theorem prover, the Isabelle/HOL system was used. Research on formalizing abstract algebra in Isabele/HOL is based on work by Hidetsune Kobayashi. This study focuses on researching mathematics, and in particular, on training researchers in the techniques of proving ([3]). In the area of mechanical theorem proving, Kobayashi gave a decision procedure for what he called abstract algebra, based on algebraic method. It is really surprise to prove many abstract algebra theorems whose traditional proofs need enormous amounts of human intelligence.

One of the key observations of Kobayashi is that theorems in abstract algebra can be relatively easily dealt with by a lot of lemmata, completely from former methods.

The power of the method can be shown by experiments on computers in which many abstract algebra theorems were proved. The success of Kobayashi's method stimulated researchers to apply the connection of lemmata images. This research on formalizing abstract algebra in Isabelle/HOL is being conducted in order to develop a CAS that supports mathematical study focused on “abstract algebra”. The system combines methods of automated theorem proving and also integrates programming in a natural way.

This method is of interest to researchers both in artificial intelligence (AI) and in algebraic modeling because they have been used in the design of programs that, in effect, can prove or disprove conjectured relationships between, or theorems about, abstract algebraic objects. It is interesting to note that theorems have been verified by this method. In a limited sense, this "theorem prover" is capable of "reasoning" about algebraic conjectures, an area often considered to be solely the domain of human intelligence.

This research aims at extending current computer systems using facilities for supporting mathematical proving. The system consists of a general higher-order predicate logic prover and a collection of special provers. The individual provers imitate the proof style of human mathematicians and produce human-readable proofs in natural language presented in nested cells. The long-term goal of this research is to produce a complete system, which supports mathematicians. On the meta-level, we can write explicit programs for reasoning tactics using Isabelle/HOL.

When researchers use the theorem prover for the acquisition of knowledge or skills, we must consider a "tool" to be a "symbol device". A symbol device exists between the researchers and the research subject. Operation activity occurs between a symbol device and the researching subject. In cognitive science, two difficulties exist, one in the interaction between the researcher and the symbol device, and one in the interaction between the symbol device and the research subject. Therefore, we must overcome these difficulties in order to effectively utilize the theorem prover in cognitive science. Moreover, we must assess the benefits of considering the integration of the theorem prover from the perspective of the relationship between mathematical knowledge and mathematical concepts. When theorem provers are used in mathematical studies, researchers achieve a result through their efforts. Then, the researchers must investigate whether conceptual problems exist or whether they simply do not appreciate how the theorem prover works. By using a theorem prover effectively, researchers become aware of numerous mathematical ideas. This is made possible by incorporating the results of research in cognitive science. In carrying out a seven-phase model of human action, "the formation of a series of intentions or actions" must be performed smoothly. The effective use of a theorem prover in cognitive science is influenced by the contents of mathematical thought, and research and understanding of mathematics can further influence general idea formation. The theorem prover influences the “perception - interpretation – evaluation” phases of evaluation. The foundations of this model were studied by Rasmussen as the three-level control model of individuals’ actions ([10]).

We can use the theorem prover as a material object that is available for the assessment of human activity. The use of the theorem prover can establish automatic and routine procedures. Controlling this automation is essential, especially in research on though processes. There are three methods for creating a theorem proof (by hand, by mind, and with a computer). A researcher's point of view of cognitive science considers the relationship between the brain and mind as the relationship between hardware and software in a computer. According to this point of view, the science of the mind is a special science, the science of thought.

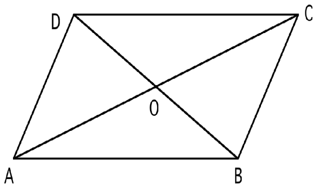
We introduce an application of CAS in Theorem proving([1]). Theorem proving uses a lot of knowledge of mathematics. Learners need active learning in mathematics education.

We taught proofs of plane geometry to learners

(University students) by using a method of Groebner basis. Then, learners showed various recognition processes. About recognition process, a lot of results are known in cognitive science. We gave the following problem, and analyzed their solving process.

Problem

Let ABCD be a parallelogram, be the intersection of the diagonals AC and BD. Show AO=OC.



The answer: The implicit assumption that the parallelogram is in a general position means that any three points among the four points A, B, C, and D can be arbitrarily chosen.

Then we can let A=(0,0), B=(u_1,0), C=(u_2,u_3), D=(x_2,x_1), and O=(x_4,x_3).

The hypothesis equations are as follows:

$h_1=u_1x_1-u_1u_3=0$ AB is parallel to DC
$h_2=u_3x_2-(u_2-u_1)x_1=0$ DA is parallel to CB
$h_3=x_1x_4-(x_2-u_1)x_3-u_1x_1=0$ O is on BD
$h_4=u_3x_4-u_2x_3=0$ O is on AC

The conclusion AO=OC is as follows:
$g=2u_2x_4+2u_3x_3-u_3^2-u_2^2=0$.

A proof by using CAS is as follows:.(We use Maple)

```
> with(Groebner):
F := [u1*x1 – u1*u3,  u3*x2 – (u2 – u1)*x1,  x1*x4 – (x2 – u1)*x3 – u1*x1, u3*x4 – u2*x3];
F := [ u1 x1 – u1 u3,  u3 x2 – (u2 – u1) x1,  x1 x4 – (x2 –u1) x3 – u1 x1, u3 x4 – u2 x3]
> G:=Basis(F, plex(x1, x2, x3, x4, u1, u2, u3))
G:= [ - u2 u1 u3 + 2 x4 u1 u3, - u3 x4 + u2 x3,  2 x3 u1 u3 – u1 u3^2, -
u2 u1 u3 + u1 u3 x2 + u1^2 u3,  2 u1 x3 x2 – 2 x3 u1^2 + 2 u1^2 u3 – u2 u1 u3,  x1 u2 – u3 x2 – u1 u3,
u1 x1 – u1 u3,  x1 x4 – x3 x2 + x3 u1 – u1 u3]
> factor(G)
[ - u1 u3 ( - 2 x4 + u2 ), - u3 x4 + u2 x3,  - u1 u3 ( - 2 x3 + u3 ),  u1 u3 ( - u2 + x2 + u1 ),  - u1 ( 2 x3 x2 – 2 x3 u1 + 2 u1 u3 – u2 u3 ),  x1 u2 – u3 x2 – u1 u3,  u1 ( x1 –  u3 ),  x1 x4 – x3 x2 + x3 u1 – u1 u3]
> with(Groebner):
 F1 := [u2 - 2*x4,  - u3*x4 + u2*x3,  - 2*x3 + u3, – u2 + x2 + u1, 2*x3*x2 – 2*x3*u1 + 2*u1*u3 – u2*u3,  x1*u2 – u3*x2 – u1*u3, x1 – u3,  x1*x4 – x3*x2 + x3*u1 – u1*u3];
F1 := [ - 2 x4 + u2,  - u3 x4 +u2 x3,  - 2 x3 + u3,
– u2 + x2 + u1,                                    2
x3 x2 – 2 x3 u1 + 2 u1 u3 – u2 u3,  x1 u2 – u3 x2 – u1 u3,  x1 – u3,  x1 x4 – x3 x2 + x3 u1 – u1 u3]
> G1:=Basis(F1, plex(x1, x2, x3, x4, u1, u2, u3))
  G1:= [ - u2 + 2 x4,  2 x3 – u3,  - u2 + x2 + u1,
x1 – u3]
> g := 2u2*x4 + 2u3*x3 – u3^2 – u2^2
  g := 2 u2 x4 + 2 u3 x3 – u3^2 – u2^2
> x3=u3/2;
> x4=u2/2;
> g;
> 0
Q.E.D.
```

This problem is a famous problem as introduction using Groebner basis for plane geometry proof.

4. Conclusion

In the three-level model of human behavior, operations and strategies can be identified and considered in relation to human thought processes in order to facilitate error-free problem solving. In consideration of surface features and conditions, similar problems can be recognized and suitable problem-solving methods can be identified. In addition, it was found that contents of the subconscious could be raised to the knowledge -based action level in order to support the expression process and the achievement of efficient functioning.

The technology of theorem prover automated reasoning. The ultimate goal of mathematics is technology. To do mathematics is gaining knowledge and solving problems by reasoning. Theorem prover is a powerful tool for researching mathematics. Researchers should appreciate the possibility of sharing cognitive level with such technology.

When it comes to making students' calculations “activity” through the introduction of technologies such as the CAS, we fear the lack of ability for the calculations. Students can become able to choose by themselves when to use the CAS. In other words, by educating in a way to make the students able to judge the use of the CAS depending on the situation, the worry on the mathematical insight will be avoided.

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