Heat Equation with Concentrated Capacity and Constant Coefficients

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Abstract-***In this paper, Dirichlet's initial-boundary value problem for the one-dimensional heat equation with concentrated capacity is considered. Eigenvalues and eigenfunctions of the corresponding spectral problem are developed for two different specific cases.***

Index Terms—*conjugation conditions, eigen-functions, eigenvalues, parabolic equations, spectral problems*

1. Introduction

LET us consider the heat equation with concen-trated capacity at interior point $x=\xi$:

$$\left[1+K\delta(x-\xi)\right]\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+f(x,t),\quad (x,t)\in(0,1)\times(0,T),\ (1)$$

with the initial value:

$$u(x,0)=u_0(x),\quad x\in(0,1),\quad (2)$$

and Dirichlet's boundary conditions:

$$u(0,t)=u(1,t)=0,\quad 0<t<T,\quad (3)$$

where $K>0$ and $\delta(x)$ is the Dirac's distribution. This problem models the heat conduction process for slim leaf with an extremely large heat capacity (see [3]). An analogous problem is considered in the paper [2].

For $(x,t)\in(0,\xi)\times(0,T)$ and $(x,t)\in(\xi,1)\times(0,T)$, the problem (1), (2), (3) can be written in the form:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+f(x,t),\ x\in(0,\xi)\cup(\xi,1),\quad t\in(0,T),\ .$$

If it is supposed that the solution is continuous on the whole interval (0,1) and if rules for differentiation of distributions are used [5], one can obtain that conditions of conjugation at point $x=\xi$ are:

$$[u]_{x=\xi}=u(\xi+0,t)-u(\xi-0,t)=0$$

$$\left[\frac{\partial u}{\partial x}\right]_{x=\xi}=K\frac{\partial u(\xi,t)}{\partial t}$$

If one puts: $H=L_2(0,1)$, $Au=-\dfrac{\partial^2 u}{\partial x^2}$, and $Bu=\left[1+K\delta(x-\xi)\right]u$, it can be easily verified that the boundary problem (1) – (3) can be expressed as an abstract Cauchy problem:

$$\boxed{B\frac{du}{dt}+Au=f(t),\quad 0<t<T,\quad u(0)=u_0}\ .$$

where u_0 is the given element from H_B, $f(t)\in L_2\left((0,T),H_{A^{-1}}\right)$ and $u(t)$ is the unknown function which transforms $(0,T)$ to H_A; H_A is the energy space defined in the usual way, with inner product $(u,v)_A$ = (Au,v). One can define H_B and $H_{A^{-1}}$ analogously (more about Hilbert spaces in [6] and [7]).

2. The Corresponding Spectral Problem

The solution of the classical initial-boundary value problem for heat equation can be expressed (see [4]) as a Fourier's series of corresponding spectral problem's eigenfunctions. In our case, that spectral problem is (see [1]):

$$\boxed{A\omega=\lambda B\omega}\ .$$

Returning to original denotations, one obtains:

$$-\frac{d^2\omega}{dx^2}=\lambda\left[1+K\delta(x-\xi)\right]\omega(x),\quad x\in(0,1)$$

$$\omega(0)=\omega(1)=0\ ,$$

or:

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$$
\begin{aligned}
&-\frac{d^2\omega}{dx^2}=\lambda\omega(x), \quad x\in(0,\xi)\cup(\xi,1)\\
&\omega(0)=\omega(1)=0\\
&[\omega]_{x=\xi}=\omega(\xi+0)-\omega(\xi-0)=0 \qquad\qquad (4)\\
&-\left[\frac{d\omega}{dx}\right]_{x=\xi}=\lambda K\omega(x)
\end{aligned}
$$

The solution of the spectral problem (4) can be written in the form:

$$
\omega(x)=\begin{cases} A\sin\alpha x, & x\in(0,\xi)\\ B\sin\alpha(1-x), & x\in(\xi,1)\end{cases}.
$$

It is obvious that $\omega(x)$ satisfies the boundary conditions.

The values of the constants A and B can be obtained by the first condition of conjugation:

$$A=\sin\alpha(1-\xi) \quad \text{and} \quad B=\sin\alpha\xi .$$

The equation $-\frac{d^2\omega}{dx^2}=\lambda\omega(x)$ gives $\lambda=\alpha^2$. Using the second condition of conjugation, one obtains:

$$\alpha=\frac{1}{K}\left[\cot\alpha(1-\xi)+\cot\alpha\xi\right], \quad (5)$$

The right hand side of the equality (5) is the sum of the two periodical functions which, in general, have different periods. So, this equation has a countable set of solutions: $\alpha=\alpha_n,\ n=1,2,\dots$, from which one can get eigenvalues: $\lambda=\lambda_n=\alpha_n^{\,2},\ 0<\lambda_1<\lambda_2<\dots<\lambda_n\to\infty$ and corresponding eigenfunctions: $\omega=\omega_n(x),\ n=1,2,\dots$.

In some cases, there exists another family of eigenvalues and corresponding eigenfunctions that is the solution of the spectral problem (4). Solutions such as this one are called "parasite solutions." Functions from this family are:

$$\omega(x)=\sin\alpha x, \quad x\in(0,l)$$

and conditions of conjugation are:

$$\omega(\xi)=0 \quad \text{and} \quad \left[\frac{d\omega}{dx}\right]_{x=\xi}=0.$$

The parasite solutions appear if ξ is rational, i.e. $\xi=\frac{p}{q}$ (one can obtain this from the second boundary condition and the conditions of conjugation). In that case $\alpha=nq\pi$, so the eigenvalues and corresponding eigenfunctions are:

$$\lambda_n=(nq\pi)^2,\quad \omega_n(x)=\sin nq\pi x,\quad n=1,2,\dots .$$

3. *Graphical Illustration*

If $\xi=\frac{1}{3}$, the equation (5) takes the form:

$$\alpha=\frac{1}{K}\left[ctg\frac{2\alpha}{3}+ctg\frac{\alpha}{3}\right].$$

In this case, the parasite solutions are:

$$\lambda_n=(3n\pi)^2,\quad \omega_n(x)=\sin 3n\pi x,\quad n=1,2,\dots .$$

The graphical solution of the equation (5) when $\xi=\frac{1}{3}$ and $K=1$ is shown in the Fig. 1. The first three eigenfunctions are presented in Fig. 2.

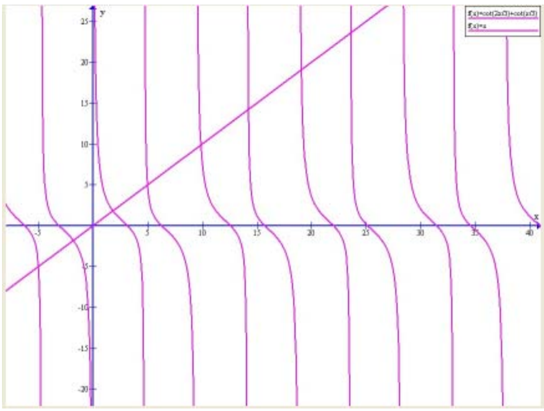

Fig. 1: The equation (5) when $\xi=\frac{1}{3}$

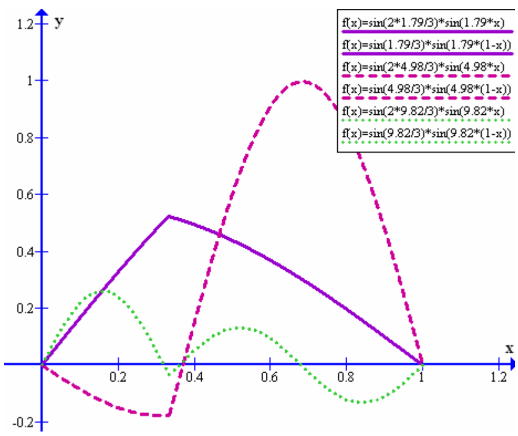

Fig. 2: The first three eigenfunctions when $\xi=\frac{1}{3}$

If $\xi=\frac{1}{\sqrt{2}}$, the equation (5) takes the form: $\alpha=ctg\alpha\left(1-\sqrt{2}\right)+ctg\sqrt{2}\alpha$.

In this case, since ξ is irrational, there are no parasite solutions.

The graphical solution of the equation (5) when $\xi = \frac{1}{\sqrt{2}}$ and $K = 1$, is shown in the Fig. 3. In the Fig. 4 the first three eigenfunctions are presented.

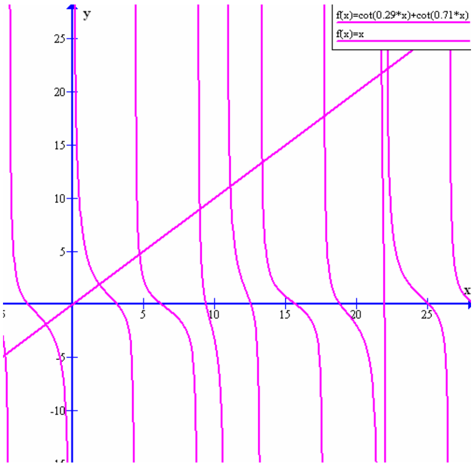

Fig. 3: The equation (5) when $\xi = \frac{1}{\sqrt{2}}$

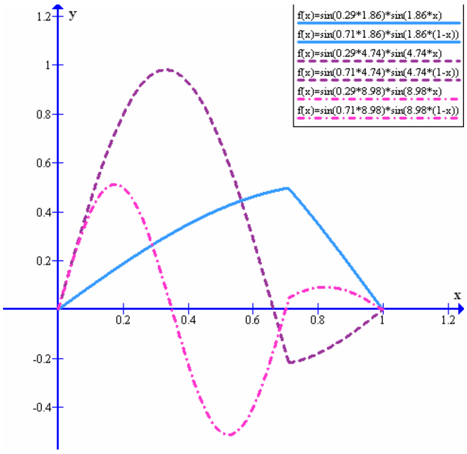

Fig. 4: The first three eigenfunctions when $\xi = \frac{1}{\sqrt{2}}$

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